2.3 Constitutive Relations

2.3.1 Definitions
- Homogeneity, Isotropy, Elasticity, Linearity
- Nonlinear Material Response

2.3.2 Generalized Hooke's Law

2.3.3 Strain Energy and Complementary Strain Energy Density Functions

2.3.4 Decomposition of Strain Energy Density Into Volumetric and Distortional Components

2.3.5 Thermal Strains and Thermal Stresses

Constitutive Relations

The analysis of stress and strains - equations of motion; and strain-displacement relationships apply to any, regardless of the material properties.

Since the response depends on the material, supplemental relations (constitutive relations) representing the type of material are needed.

Constitutive relations are semi-empirical: based on experimental observation.

Relations between stress components

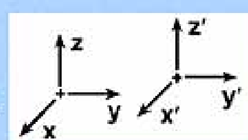
$$\left\| \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{yz}, \tau_{zx}, \tau_{xy} \right\|$$

and strain components $\left\| \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy} \right\|$

Definitions

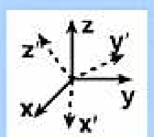
Homogeneity

A material property is called homogeneous if it does not change from point to point in the body (i.e., it is invariant under coordinate translation).



Isotropy

A material property is called isotropic if it does not change with direction (i.e., it is invariant under coordinate rotation).



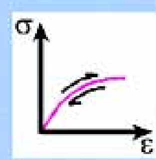
Definitions

Isotropy

A material property is called isotropic if it does not change with direction (i.e., it is invariant under coordinate rotation).

Elasticity

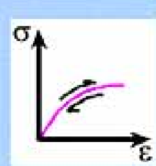
The material is called elastic if its loading and unloading curves coincide.



Definitions

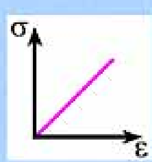
Elasticity

The material is called elastic if its loading and unloading curves coincide.



Linearity

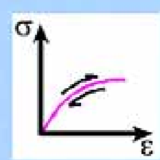
Refers to linear dependence of stresses on strains.

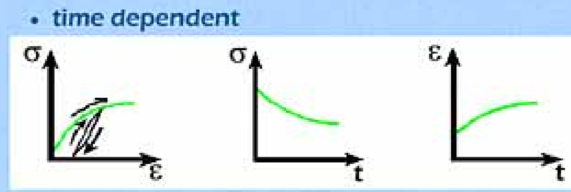


Definitions

Nonlinear Material Response

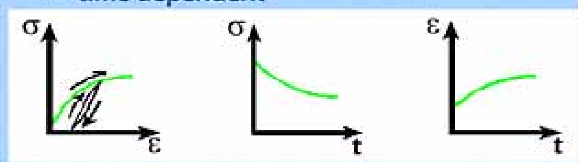
- Nonlinear elastic response
  - single-valued relationship between stresses and strains
- Inelastic response
  - time independent
  - time dependent





Definitions

- Inelastic response
  - time independent
  - time dependent



Generally, more than one material model is needed for the entire stress-strain-temperature range of interest.

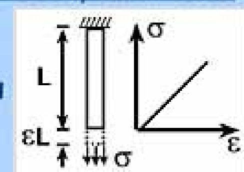
Generalized Hooke's Law

Hooke's Law for One Dimensional Stress / Strain State

$$\sigma = E\,\varepsilon$$

where σ, ε = uniaxial mechanical stress and strain

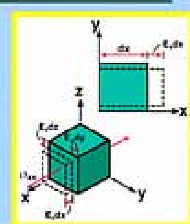
E = Young's modulus

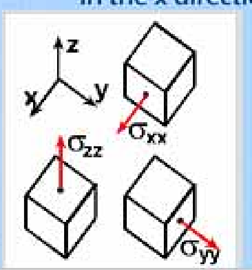


Generalized Hooke's Law

Generalized Hooke's Law

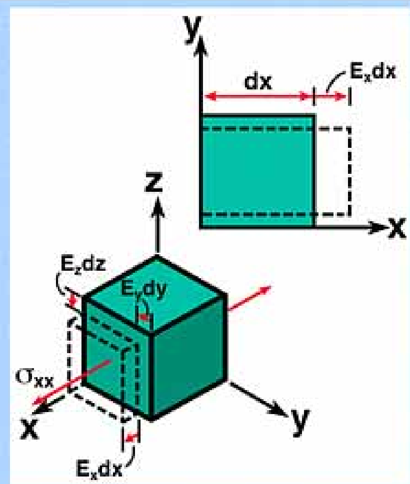
- For linearly elastic isotropic material
  - Linear (extensional) strain in the x direction, ε_x



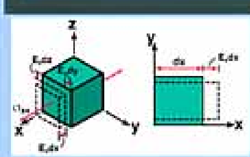


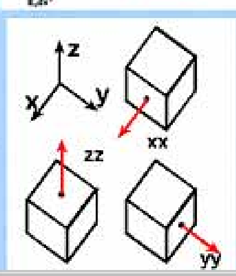
associated with

$$\sigma_{xx} : \frac{1}{E}\sigma_{xx}$$
$$\sigma_{yy} : -\frac{\nu}{E}\sigma_{yy}$$
$$\sigma_{zz} : -\frac{\nu}{E}\sigma_{zz}$$



Generalized Hooke's Law





$$\sigma_{xx} : \frac{1}{E}\sigma_{xx}$$
$$\sigma_{yy} : -\frac{\nu}{E}\sigma_{yy}$$
$$\sigma_{zz} : -\frac{\nu}{E}\sigma_{zz}$$

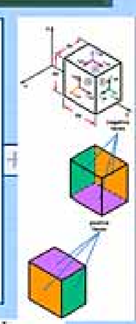
$$\varepsilon_x = \frac{1}{E}\left(\sigma_{xx} - \nu\,\sigma_{yy} - \nu\,\sigma_{zz}\right)$$

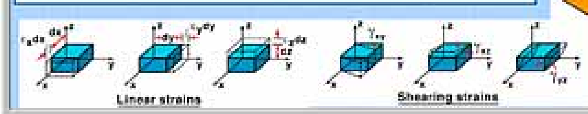
Shearing strain in plane xy

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

Generalized Hooke's Law

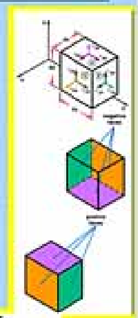
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \cdot & \cdot & \cdot \\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} & \cdot & \cdot & \cdot \\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \frac{1}{G} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{G} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{G} \end{array}\right] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

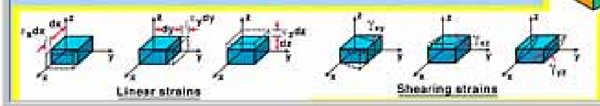




Generalized Hooke's Law

where ν = Poisson's ratio,
G = Shear modulus

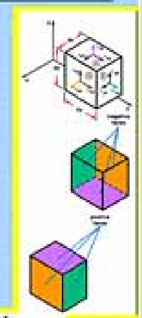


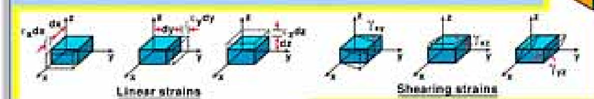


Generalized Hooke's Law

Comments on Generalized Hooke's Law

- For linearly elastic isotropic materials
  - Extensional strains are associated with normal stresses
  - Shearing strain in each of the coordinate planes is associated with the shearing stress on the same plane

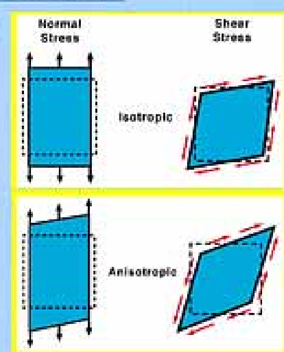


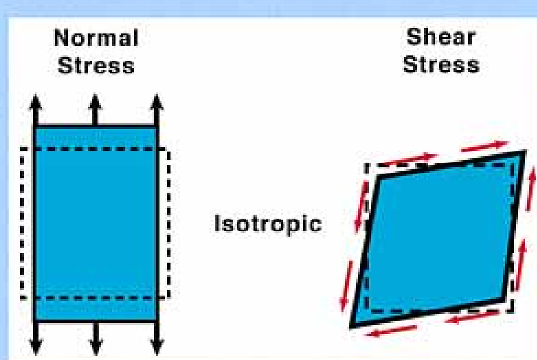


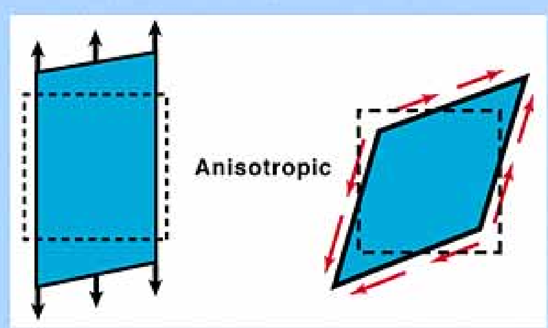
Generalized Hooke's Law

Comments on Generalized Hooke's Law

- Two independent material coefficients
  - E = Young's modulus
  - υ = Poisson's ratio
  - G = shear modulus $= E / 2(1+\upsilon)$
- Anisotropic mechanical properties
  - extensional and shear effects are coupled



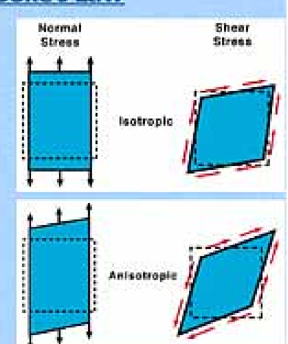




Generalized Hooke's Law

Comments on Generalized Hooke's Law

- Anisotropic mechanical properties
  - extensional and shear effects are coupled
- For nonhomogeneous materials, E and ν are functions of the coordinates.



Generalized Hooke's Law

- The relations shown apply for the case of strains and stresses caused by mechanical loading (not for thermal, magnetic and/or electric fields).

Adding the first three equations:

$$(\varepsilon_x + \varepsilon_y + \varepsilon_z) = \frac{1-2\upsilon}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$= \frac{1}{3K}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

where $K = \dfrac{E}{3(1-2\upsilon)}$ = bulk modulus

or $\left(\frac{1}{3}J_1\right) = \frac{1}{3K}\left(\frac{1}{3}I_1\right)$

Generalized Hooke's Law

$$(\varepsilon_x + \varepsilon_y + \varepsilon_z) = \frac{1-2\upsilon}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$= \frac{1}{3K}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

where $K = \dfrac{E}{3(1-2\upsilon)}$ = bulk modulus

or $\left(\frac{1}{3}J_1\right) = \frac{1}{3K}\left(\frac{1}{3}I_1\right)$

which is a relation between the volumetric strain and volumetric stress components.

Strain Energy and Complementary Strain Energy Density Functions

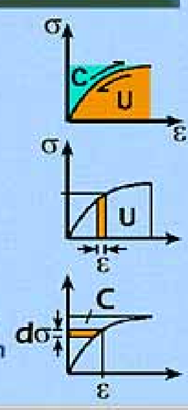
For elastic materials and uniaxial stress state

$$\sigma\,\varepsilon = U + C$$

U = strain energy density (strain energy per unit volume)

$$= \int \sigma\, d\varepsilon$$

C = complementary strain energy density (complementary strain energy per unit volume)



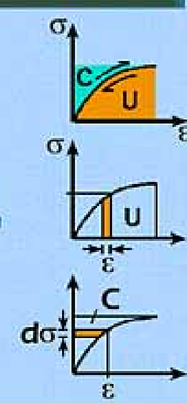
Strain Energy and Complementary Strain Energy Density Functions

U = strain energy density (strain energy per unit volume)

$$= \int \sigma\, d\varepsilon$$

C = complementary strain energy density (complementary strain energy per unit volume)

$$= \int \varepsilon\, d\sigma$$



Strain Energy and Complementary Strain Energy Density Functions

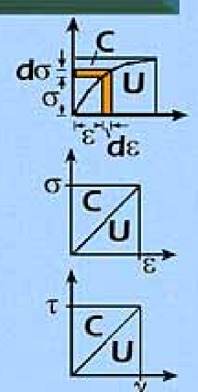
From which

$$\sigma = \frac{dU}{d\varepsilon} \qquad \varepsilon = \frac{dC}{d\sigma}$$

- For linearly elastic materials

$$U = \frac{1}{2}E\,\varepsilon^2 \qquad C = \frac{1}{2E}\sigma^2$$

- For the case of pure shear - linearly elastic materials



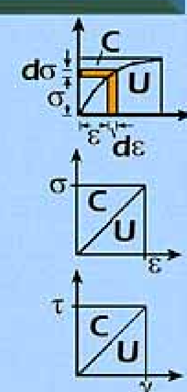
Strain Energy and Complementary Strain Energy Density Functions

- For linearly elastic materials

$$U = \frac{1}{2}E\,\varepsilon^2 \qquad C = \frac{1}{2E}\sigma^2$$

- For the case of pure shear - linearly elastic materials

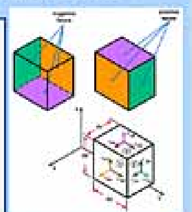
$$U = \frac{1}{2}G\,\gamma^2 \qquad C = \frac{1}{2G}\tau^2$$



Strain Energy and Complementary Strain Energy Density Functions

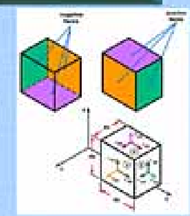
- For the three-dimensional stress state, the strain energy density, and the complementary strain energy density are given by:

$$dU = \left[\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \middle|\ \tau_{yz}\ \tau_{zx}\ \tau_{xy}\right]\left[\begin{array}{c} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_z \\ \hline d\gamma_{yz} \\ d\gamma_{zx} \\ d\gamma_{xy} \end{array}\right]$$



Strain Energy and Complementary Strain Energy Density Functions

$$dC = \left[\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \middle|\ \gamma_{yz}\ \gamma_{zx}\ \gamma_{xy}\right]\left[\begin{array}{c} d\sigma_{xx} \\ d\sigma_{yy} \\ d\sigma_{zz} \\ \hline d\tau_{yz} \\ d\tau_{zx} \\ d\tau_{xy} \end{array}\right]$$

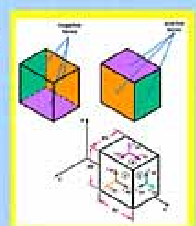


Strain Energy and Complementary Strain Energy Density Functions

- Total strain energy and total complementary strain energy are given by:

$$\bar{U} = \int_V U\, dV$$

$$\bar{C} = \int_V C\, dV$$

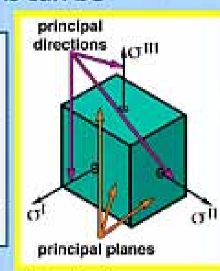


Decomposition of Strain Energy Density into Volumetric and Distortional Components

- The principal stresses and strains can be decomposed as follows:

$$\begin{Bmatrix} \sigma^{I} \\ \sigma^{II} \\ \sigma^{III} \end{Bmatrix} = \underbrace{\frac{1}{3}\begin{Bmatrix} I_1 \\ I_1 \\ I_1 \end{Bmatrix}}_{\text{volumetric}} + \underbrace{\begin{Bmatrix} \sigma^{I} - \frac{1}{3} I_1 \\ \sigma^{II} - \frac{1}{3} I_1 \\ \sigma^{III} - \frac{1}{3} I_1 \end{Bmatrix}}_{\text{deviatoric}}$$

where $I_1 = \sigma^{I} + \sigma^{II} + \sigma^{III}$

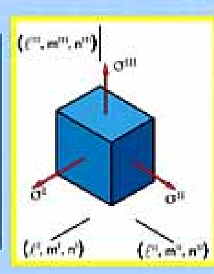


Decomposition of Strain Energy Density into Volumetric and Distortional Components

Analogously,

$$\begin{Bmatrix} \varepsilon^{I} \\ \varepsilon^{II} \\ \varepsilon^{III} \end{Bmatrix} = \underbrace{\frac{1}{3}\begin{Bmatrix} J_1 \\ J_1 \\ J_1 \end{Bmatrix}}_{\text{volumetric}} + \underbrace{\begin{Bmatrix} \varepsilon^{I} - \frac{1}{3} J_1 \\ \varepsilon^{II} - \frac{1}{3} J_1 \\ \varepsilon^{III} - \frac{1}{3} J_1 \end{Bmatrix}}_{\text{deviatoric}}$$

where $J_1 = \varepsilon^{I} + \varepsilon^{II} + \varepsilon^{III}$

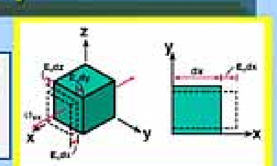


Decomposition of Strain Energy Density into Volumetric and Distortional Components

For linearly elastic materials

$$U = \frac{1}{2}\left(\sigma^{I}\varepsilon^{I} + \sigma^{II}\varepsilon^{II} + \sigma^{III}\varepsilon^{III}\right)$$

$$= U_{vol.} + U_{dist.}$$

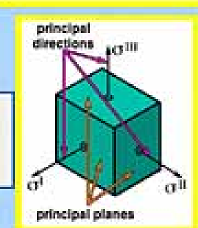


where $U_{vol.} = \frac{1}{6} I_1 J_1$

but, for isotropic materials

$$\begin{Bmatrix} \varepsilon^{I} \\ \varepsilon^{II} \\ \varepsilon^{III} \end{Bmatrix} = \frac{1}{E}\begin{vmatrix} 1 & -\upsilon & -\upsilon \\ -\upsilon & 1 & -\upsilon \\ -\upsilon & -\upsilon & 1 \end{vmatrix}\begin{Bmatrix} \sigma^{I} \\ \sigma^{II} \\ \sigma^{III} \end{Bmatrix}$$

$$J_1 = \frac{1}{3K} I_1$$

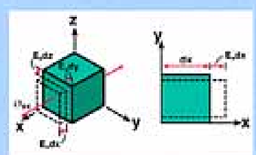


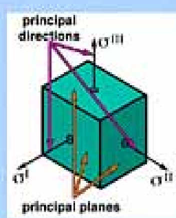
Decomposition of Strain Energy Density into Volumetric and Distortional Components

Therefore,

$$U = \frac{1}{2E}\left[(\sigma^{I})^2 + (\sigma^{II})^2 + (\sigma^{III})^2 - 2\upsilon\left(\sigma^{I}\sigma^{II} + \sigma^{II}\sigma^{III} + \sigma^{III}\sigma^{I}\right)\right]$$

$$U_{vol.} = \frac{1}{18K}(I_1)^2$$



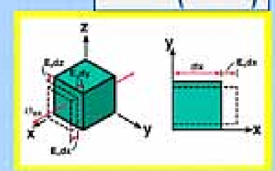


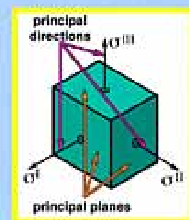
Decomposition of Strain Energy Density into Volumetric and Distortional Components

Therefore,

$$U_{dist.} = U - U_{vol.}$$
$$= \frac{1}{12G}\left[(\sigma^{I} - \sigma^{II})^2 + (\sigma^{II} - \sigma^{III})^2 + (\sigma^{III} - \sigma^{I})^2\right]$$

where $G = \dfrac{E}{2(1+\upsilon)}$

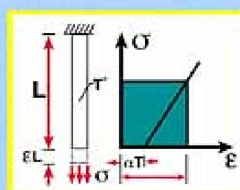




Thermal Strains and Thermal Stresses

Hooke's Law for One Dimensional Stress / Strain State

$$\boxed{\sigma = E(\varepsilon - \alpha T)}$$



where

ε is the total strain

αT is the thermal strain

σ is related to the mechanical strain